Lec 1.1. Functions of Several Variables, Level Curves, and Level Surfaces

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Objectives

- To identify and sketch the natural domain of functions of several variables
- To visualize functions of two/three variables using level curves/surfaces and contour plots

Introduction

Recall: A **function** f from a set X to a set Y, indicated by the notation

$$f : X \to Y$$

is an object that pairs each element in X to an element in Y.

Q: What do we call these sets X and Y?

Introduction

$$\boxed{f : X \to Y}$$

For the first part of Math 23, we are interested in the case where $Y = \mathbb{R}$ and X is a subset of $\mathbb{R}^n$.

Definition

$\mathbb{R}^n$ is the set of objects of the form $(x_1, x_2, \ldots, x_n)$, where each of the $x_j' s$ are real numbers. Essentially, they are lists of n real numbers.

Example

1. The set $\mathbb{R}^2$ consists of ordered pairs (x, y) of real numbers (ex. $(0, 1), (\pi, \sqrt{2})$).
2. The set $\mathbb{R}^4$ consists of ordered *quadruples* (w, x, y, z) of real numbers. (ex. $(-1, 3, 2.5, 1)$).
3. The set

$$\{(x, y, z) \mid x, y, z \in \mathbb{R}, x^2 + 2y^2 + 3z^2 = 1\}$$

is a subset of $\mathbb{R}^3$. (Q: What is special about this set?)

$$\boxed{f : X \to \mathbb{R}}$$

If the domain X is a subset of $\mathbb{R}^n$, we say f is a **(real-valued) function of n (real) variables**.

1. If $n = 1$, we often write f as "$y = f(x)$".
2. If $n = 2$, f can also be expressed as "$z = f(x, y)$".
3. If $n = 3$, f may be written in the form "$w = f(x, y, z)$".

Example

1. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
$$f(x, y) = x^2 + y^2$$
is a function of 2 variables. Q: What is $f(2, 3)$?

2. The function $g : X \to \mathbb{R}$, where
$$X = \{(x, y, z) \in \mathbb{R}^3 \mid y < x^2\},$$
$$g(x, y, z) = \frac{z}{\sqrt{x^2 - y}},$$
is a function of 3 variables. Q: What is $g(2, 3, 1)$?

Technically, the domain of a function must be indicated *before* describing the actual function.

In practice, we are given a description of the function first, and it is up to us to assign a **suitable domain** to that function (often, it is the largest set of valid inputs).

Example

Find and sketch the domains of the following functions.

1. $f(x, y) = \ln(1 - x^2 - y^2)/(x^2 + y^2)$
2. $g(x, y) = \sqrt{y - x} \cdot \ln(x + y)$
3. $h(x, y, z) = xz \sin^{-1}(1 - y^2)$

Remarks

1. Use <u>circles</u> to indicate that a point is not part of the domain being sketched.
2. Use <u>dashed curves</u> to indicate that a line/boundary of a region is not part of the domain.
3. Recall the domains of various special functions:
   - $\operatorname{dom}\ln = (0, \infty)$
   - $\operatorname{dom}\sqrt[n]{\cdot} = [0, \infty)$ if n is <u>even</u>, $\mathbb{R}$ if n is <u>odd</u>
   - $\operatorname{dom}\sin^{-1} = \operatorname{dom}\cos^{-1} = [-1, 1]$

(*Note:* "dom" = *domain*)

Q: How do we visualize functions of n-variables?

Recall: To visualize the single-variable function $y = f(x)$, we consider the set

$$\{(x, f(x)) \mid x \in \operatorname{dom} f\}$$

known as the **graph** of f, then plot this in $\mathbb{R}^2$ (via the 2D Cartesian coordinate system).

If f is a function of two variables, say $z = f(x, y)$, then we may consider the set

$$\{(x, y, f(x, y)) \mid (x, y) \in \operatorname{dom} f\}$$

also known as the **graph** of f, then plot this in $\mathbb{R}^3$ (the 3D Cartesian coordinate system).

Example

Consider the function $f(x, y) = \sqrt{2x^2 + y^2}$.

Q: Can the same approach be used to visualize functions of three variables or more?

Producing graphs for functions of three or more variables is difficult since $\mathbb{R}^n$ has no easy visualization for $n \geq 4$.

However, there is an alternative for functions of two and three variables.

Definition

Let f and g be functions of two and three variables, resp., and let k be a real number.

1. The **level curve** of f at k is the set

$$\{(x, y) \in \operatorname{dom} f \mid f(x, y) = k\}$$

2. The **level surface** of g at k is the set

$$\{(x, y, z) \in \operatorname{dom} g \mid g(x, y, z) = k\}$$

A collection of level curves/surfaces is called a **contour plot**.

Example

1. Plot the level curves of $f(x, y) = x^2 - y$ at $k = 0, 1, 2, 3$.
2. Plot the level curves of $g(x, y) = y^2 - x^2$ at $k = -4, -2, 0, 2, 4$.
3. Plot the level surfaces of $h(x, y, z) = x^2 + y^2 + z^2$ at $k = 0, 1, 9$.

Remarks

- A level curve is essentially a "slice" of a function of two-variables that is parallel to the xy-plane.
- Given enough of these slices, we can get a rough picture of how the function behaves/appears.
- Topographic maps are essentially contour plots. If $z = f(x, y)$ models the height of land at latitude x and longitude y, a topographic map of the land is a contour plot of f with many level curves.
- Level surfaces allow us to see how a function of three-variables changes in value.

Seatwork 1

1. Identify and sketch the domain of the function

$$f(x, y) = \frac{\ln(x - y)}{\sqrt{4 - x^2 - y^2}}$$

2. Sketch a contour map of the function $p(x, y) = \dfrac{y}{x}$ for $k = 1, 2, 3.$