Lec 1.2. Limits and Continuity of Functions of Several Variables

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Objectives

- To know how the familiar concepts of "limit" and "continuity" is extended to functions of two variables
- To know how to disprove the existence of limits using the idea of "limits along curves"

Introduction

Recall: Let f be a function defined on an interval I containing the real number a. Intuitively, we say

$$\lim_{x \to a} f(x) = L$$

provided the output $f(x)$ gets closer in value to L as the input x gets closer to a.

Q: How can we extend this to two-variable functions?

Before: Let f be a function defined on an interval I containing the real number a. Intuitively, we say

$$\lim_{x \to a} f(x) = L$$

provided the output $f(x)$ gets closer in value to L as the input x gets closer to a.

After: Let f be a two-variable function defined on a region in $\mathbb{R}^2$ containing the point (a, b). Intuitively, we say

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

provided the output $f(x, y)$ gets closer in value to L as the input (x, y) gets closer to (a, b).

" Let f be a two-variable function defined on a **region** in $\mathbb{R}^2$ containing the point (a, b). Intuitively, we say

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

provided the output $f(x, y)$ gets **closer** in value to L as the input (x, y) gets **closer** to (a, b)."

Concerns:

- How do we describe this region?
- How do we measure closeness?

Definition

1. The **distance** between two *numbers* x and y in $\mathbb{R}$ is, by convention, measured as $|x - y|$.
2. The **distance** between two *points* (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$ is, by convention,
$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$
3. An **open disk** in $\mathbb{R}^2$ is a disk without boundary. Formally, an open disk centered at (a, b) with radius $r > 0$ is the set
$$\{(x, y) \in \mathbb{R}^2 \mid \underbrace{\sqrt{(x-a)^2 + (y-b)^2} < r}_{\textit{dist. bet. } (x,y) \textit{ and } (a,b) \textit{ is } < r}\}.$$

Before: Let f be a two-variable function defined on a region in $\mathbb{R}^2$ containing the point (a, b). Intuitively, we say

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

provided the output $f(x, y)$ gets closer in value to L as the input (x, y) gets closer to (a, b).

After: Let f be a two-variable function defined on an open disk D containing the point (a, b). Intuitively, we say

$$\lim_{(x,y)\to(a,b)} f(x) = L$$

provided $|f(x) - L|$ gets smaller as $\sqrt{(x-a)^2 + (y-b)^2}$ gets smaller.

"$|f(x) - L|$ gets smaller as $\sqrt{(x-a)^2 + (y-b)^2}$ gets smaller"

$$\Longleftrightarrow$$

$|f(x) - L|$ can be made as small as we want so long as $\sqrt{(x-a)^2 + (y-b)^2}$ is small enough

$$\Longleftrightarrow$$

for any small number $\epsilon > 0$, $|f(x) - L| < \epsilon$ so long as $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ for some small number δ

Definition (due to Cauchy)

Let f be a two-variable function defined on an open disk D containing the point (a, b). We say that the **limit** of f as (x, y) approaches (a, b) is $L \in \mathbb{R}$, with notation

$$\boxed{\lim_{(x,y)\to(a,b)} f(x, y) = L}$$

provided that for any $\epsilon > 0$, there exists $\delta > 0$ for which

$$|f(x, y) - L| < \epsilon \text{ whenever } \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

Remark

- Understanding the definition of the limit is one thing. **Using it to prove limits is another**. (A math major must be able to do both!)
- In this course you will not be asked to prove limits using the formal definition.

There are functions whose limits can be easily obtained. In Math 21, these are the continuous functions.

Definition

The two-variable function f is said to be **continuous** at the point (a, b) provided that:

1. $f(a, b)$ exists (i.e. (a, b) is in the domain of f),
2. $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists, and
3. the two quantities are equal.

Otherwise, we say f is **discontinuous** at (a, b).

Furthermore, f is said to be **continuous on the region** $R \subseteq \mathbb{R}^2$ if f is continuous at each point in R.

Remarks (Part 1)

1. In essence, taking the limit of a continuous function is the same as evaluating it.
2. All polynomial functions in the variables x and y are continuous.
3. Limits in Math 23 inherit the same familiar properties of limits from Math 21.

Remarks (Part 2)

For the two-variable functions f and g such that $\lim f(x, y) = L$ and $\lim g(x, y) = M$, the following are true:

a. $\lim[f(x, y) \pm g(x, y)] = \lim f(x, y) \pm \lim g(x, y) = L + M$
b. $\lim[f(x, y) \cdot g(x, y)] = \lim f(x, y) \cdot \lim g(x, y) + L \cdot M$
c. $\lim[f(x, y) \div g(x, y)] = \lim f(x, y) \div \lim g(x, y) = L/M$, so long as $\lim g(x, y) \neq 0$.
d. Given a function $h(u)$, if $\lim\limits_{u \to L} h(u)$ exists, then

$$\lim_{(x,y)\to(a,b)} h(f(x, y)) = \lim_{u \to L} h(u).$$

To analyze limits for single-variable functions in Math 21, we used one-sided limits as a tool.

Q: Is there an analogue for two-variable functions?

On $\mathbb{R}$, there are essentially two paths (i.e. from the left/right) you can take in order to approach a number. But on $\mathbb{R}^2$, there are **infinitely many paths** we can use to approach a point!

Definition (Part 1)

Let h be a two-variable function defined on an open disk containing (a, b).

Let C_1 be the curve defined by the graph of the function $y = f(x)$ that passes through (a, b) (so, $b = f(a)$).

We define the **limit of h as (x, y) approaches (a, b) along the curve C_1** by

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{along } C_1}} h(x, y) := \lim_{x\to a} h(x, f(x)).$$

Definition (Part 2)

Similarly, let C_2 be the curve defined by the graph of the function $x = g(y)$ that passes through (a, b) (so $a = g(b)$).

We define the **limit of** h **as** (x, y) **approaches** (a, b) **along the curve** C_2 **by**

$$\lim_{\substack{(x,y)\to(a,b) \\ \text{along } C_1}} h(x, y) := \lim_{y\to b} h(g(y), y).$$

Example

Consider the function

$$h(x, y) = \frac{x + y^2 - 1}{2(x - 1)^2 + y^2}$$

Evaluate the limit of h as (x, y) approaches $(1, 0)$:

1. along the curve $C_1 : y = x - 1$
2. along the curve $C_2 : x = 1$

In Math 21, we could prove that the limit of a function does not exist by showing that the limits from the left and right are not the same.

Here in Math 23, we can prove that limits do not exist by showing that the limits **along two different paths** are not equal. (Q: Why?)

Example

Show that the following limits do not exist.

1. $$\lim_{(x,y)\to(0,0)} \frac{2x^2 - y^2}{x^2 + y^2}$$
2. $$\lim_{(x,y)\to(2,0)} \frac{2(x-2)y}{(x-2)^2 + y^2}$$

Remarks

1. We only use limits along curves to prove **nonexistence of limits**. You cannot use these to prove existence.
2. When choosing a curve, always check if it really passes through the point being investigated.
3. Choosing two appropriate curves takes practice.
(Start with lines, i.e. $x = a$, $y = b$)
4. Knowing how to disprove limits allows us to check for discontinuities present in a function.

Definition

Suppose $f(x, y)$ is discontinuous at the point (a, b).

- If $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, we say that the discontinuity is **removable**.
- Otherwise, we say that the discontinuity is **essential**.

Example

Consider the function

$$f(x, y) = \begin{cases} \dfrac{3xy^2}{2x^2 + y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Is f continuous at $(0, 0)$? If not, what kind of discontinuity is present?

Example (Try it yourself!)

Consider the function

$$f(x,y) = \begin{cases} \dfrac{1 - e^{-2x^2-2y^2}}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$$

Is f continuous at $(0,0)$? If not, what kind of discontinuity is present?