Lec 1.4. Local Linear Approximation, Differentials, and Differentiability

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Objectives

- To know how to compute the local linear approximation of a function at a point
- To know how to estimate changes in function values due to small input deviations through differentials
- To determine the conditions for a local linear approximation to be a good approximation

Recall: Let f be a single-variable function. Suppose we have information at $x = a$: $f(a)$ and $f'(a)$. If x is near a, then

$$f(x) \approx f'(a)(x - a) + f(a)$$

Essentially, if we have information on f at $x = a$, then we can estimate the values of f near $x = a$ using a linear function.

Geometrically, this linear function is the tangent line at $x = a$.

Q: How to extend this to functions of several variables (say, two)?

If **lines** can be used to approximate <u>curves</u>, then ________ can be used to approximate <u>surfaces</u>. Ans: **planes**

Let us use planes to approximate graphs of two-variable functions (since they are surfaces)!

Recall: equation of a plane passing through the point (a, b, c):

$$A(x-a)+B(y-b)+C(z-c)=0$$

By isolating z, we get

$$z=\underbrace{\tilde{A}(x-a)+\tilde{B}(y-b)+\tilde{C}}_{\text{plane as function of } x \text{ and } y}$$

Goal: Given a function $z = f(x, y)$ and a point (a, b) in dom f, find constants A, B and c such that

$$f(x, y) \approx A(x - a) + B(y - b) + c$$

when (x, y) is near (a, b).

We will denote the right side by $L_{(a,b)}(x, y)$.

Idea: The left and right side should have the same "properties" at the point (a, b). In particular, let us force:

1. $f(a, b) = L_{(a,b)}(a, b)$
2. $f_x(a, b) = \partial_x L_{(a,b)}(a, b)$
3. $f_y(a, b) = \partial_y L_{(a,b)}(a, b)$

Definition

The **local linear approximation (LLA)/linearization** of a two-variable function f at (a, b) is defined as

$$\boxed{L_{(a,b)}(x, y) = \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b) + f(a, b)}$$

It is an estimate for $f(x, y)$ when (x, y) is near (a, b).

Remark

1. It is defined using linear functions, hence "linear".
2. For $\overset{\text{(not all!)}}{\textit{many}}$ functions, the LLA gives good approximations.
3. Often, it is the tangent plane to the graph of $z = f(x, y)$ at $(a, b, f(a, b))$.

Example

1. Find the linearization/LLA of $f(x, y) = xy^2 + y\cos(x-1)$ at $(1, 2)$ and use this to estimate the value of $f(1.1, 1.9)$.
2. Find the LLA of $g(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate $g(1.95, 1.08)$.
3. Determine the equation of the tangent plane to the surface of the graph of $h(x, y) = x\sin(xy)$ at the point $P(1, \frac{\pi}{2}, 1)$.

Remark

In general, if f depends on the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, then the local linear approximation to $f(\mathbf{x})$ when $\mathbf{x}$ is near the point $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is

$$L_{\mathbf{a}}(\mathbf{x}) = \frac{\partial f}{\partial x_1}(\mathbf{a})(x_1 - a_1) + \frac{\partial f}{\partial x_1}(\mathbf{a})(x_2 - a_2) + \ldots$$

$$\ldots + \frac{\partial f}{\partial x_n}(\mathbf{a})(x_n - a_n) + f(\mathbf{a}).$$

Example

Find the linearization of the function $w(x, y, z) = xye^{xz}$ at $(1, 1, 0)$.

We can use LLAs/linearization to address a common real world problem:

How does the output of a function change with respect to slight changes on its input?

Suppose f is a two-variable function with input (x, y). Consider the slightly modified input $(x + dx, y + dy)$, where dx and dy are small numbers. Let us estimate:

$$f(x + dx, y + dy) - f(x, y).$$

Using LLA/linearization, when (x', y') is near (x, y),

$$f(x', y') \approx f_x(x, y)(x' - x) + f_y(x, y)(y' - y) + f(x, y).$$

Since $(x + dx, y + dy)$ is near (x, y):

$$\begin{aligned} & f(x + dx, y + dy) \\ & \approx f_x(x, y)[x + dx - x] + f_y(x, y)[y + dy - y] + f(x, y) \\ & = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy + f(x, y) \end{aligned}$$

Therefore,

$$\boxed{f(x + dx, y + dy) - f(x, y) \approx f_x(x, y) \cdot dx + f_y(x, y) \cdot dy}$$

Definition

The **(total) differential** of the two-variable function f, denoted by df, at (x, y) is the expression

$$\boxed{df(x, y) = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy}$$

It estimates the change in value to f when the input (x, y) is changed to $(x + dx, y + dy)$.

Example

1. Using differentials, estimate the amount of tin in a closed tin can with inner diameter 8 cm and inner height 12 cm if the tin is 0.04 cm thick.

Example

2. A boundary strip 3 in wide is painted around a rectangle whose dimensions are 100 ft by 200 ft. Use differentials to approximate the amount (in square feet) of paint needed to cover the strip.
3. A steel factory produces a canister by creating an open cylinder of radius 3 in and height 10 in then covering both ends with two hemispheres. To ensure a precise volume, which dimension must be more closely monitored?

Differentials can be used to assess the sensitivity of a function to small input changes.

For multivariable functions, to estimate the change in value to f when the input $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is changed to $(x_1 + dx_1, x_2 + dx_2, \ldots, x_n + dx_n)$, we use the differential :

$$\boxed{df(\mathbf{x}) = f_{x_1}(\mathbf{x}) \cdot dx_1 + f_{x_2}(\mathbf{x}) \cdot dx_2 + \ldots + f_{x_n}(\mathbf{x}) \cdot dx_n}$$

Example

A factory is tasked to produce a cardboard box of dimensions 10 cm $\times$ 20 cm $\times$ 6 cm. Due to a manufacturing error, the box produced is 0.1 cm shorter than what was asked. Estimate the change in volume of the box due to this mistake.

As mentioned earlier, the LLA gives good local approximations for *many* functions, but not all.

Example

Find the LLA of the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 - y^2}, & \text{if } x^2 - y^2 \neq 0 \\ 0, & \text{if } x^2 - y^2 = 0. \end{cases}$$

at the point $(0,0)$. Compare the LLA and the actual function value at the points $(x,y) = (0.1, 0.09)$ and $(-0.001, 0.00099)$.

Definition

A function f of the variables x and y is said to be **differentiable** at (a, b) provided that

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y) - L_{(a,b)}(x,y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0.$$

Essentially, the LLA error near (a, b) has to vanish faster than the rate at which the input gets closer to (a, b).

$$\boxed{\textbf{diff'ble at } (a,b) \iff \textbf{LLA is good enough near } (a,b)}$$

More on Differentiability

For a function of two variables $f(x, y)$:

1. If f_x and f_y are continuous on an open disk D (or $\mathbb{R}^2$), then f is differentiable at each point in D (or $\mathbb{R}^2$). (Many functions satisfy this.)
2. f differentiable at $(a, b) \Rightarrow$ well-defined tangent plane at $(a, b, f(a, b))$
3. f differentiable at $(a, b) \Rightarrow f$ continuous at (a, b).

The above results can be extended to general multivariable functions.

Example

In the previous problem:

"Find the linearization/LLA of $f(x, y) = xy^2 + y\cos(x-1)$ at $(1, 2)$ and use this to estimate the value of $f(1.1, 1.9)$."

Justify the usage of the LLA by showing that f is differentiable at $(1, 2)$.

Seatwork 3

1. Find the linearization of $f(x, y) = \ln(3x - 2y)$ at the point $(2, 2)$ and use this to estimate the value of
$$\ln(3(2.01) - 3.98).$$
2. A cardboard manufacturer was asked to produce a box with dimensions 30 in $\times$ 25 in $\times$ 10 in. Due to a manufacturing error, the box produced has dimensions 30.2 in $\times$ 25.4 in $\times$ 9.8 in. Use differentials to estimate how much the surface area of the box has deviated from its ideal measure.