Lec 1.5. Multivariable Chain Rule and Implicit Differentiation

Peter John M. Burgos

Institute of Mathematics, UP Diliman

Objectives

- To know how the chain rule extends to multivariable functions
- To know how to differentiate implicitly-defined functions

Recall: Suppose that the function f depends on the quantity u, and in turn u depends on the variable x (i.e. $u = u(x)$).

$$f \to u \to x$$

Then f can be viewed as a function of x, and by the **chain rule**:

$$\boxed{\frac{d\left(f(u)\right)}{dx} = \left.\frac{df}{du}\right|_{u=u(x)} \cdot \frac{du}{dx}}$$

Example

Consider the function $f(x) = \sin(2x^2 + x)$. This can be viewed as the function

$$f(u) = \sin(u)$$

where u, in turn, is dependent on the variable x by the equation $u = 2x^2 + x$. Thus, we can think of f as a function of x.

Q: What is $\frac{df}{dx}$?

In general, it is possible for a function to depend on several quantities and, in turn, these quantities can be made dependent on several variables.

Case 1.

Suppose f depends on two quantities: x and y. Further suppose each of these quantities depend smoothly on the variable t. Q: What is $\frac{df}{dt}$?

$$\begin{tikzcd} & f \arrow[dl] \arrow[dr] & \\ x \arrow[d] & & y \arrow[d] \\ t & & t \end{tikzcd}$$

$$\frac{d}{dt}f(x(t),y(t)) = \lim_{\Delta t\to 0}\frac{f(x(t+\Delta t),y(t+\Delta t)) - f(x(t),y(t))}{\Delta t}.$$

Ideas:

1. For small Δt, $x(t+\Delta t) \approx x(t) + x'(t)\Delta t$.
2. For small Δt, $y(t+\Delta t) \approx y(t) + y'(t)\Delta t$.
3. Using differentials,

$$f(x+dx, y+dy) - f(x,y) \approx f_x(x,y)\,dx + f_y(x,y)\,dx,$$

which gets more accurate as $dx, dy \to 0$.

Using these ideas, we can conclude

$$\boxed{\frac{df}{dt} = \left.\frac{\partial f}{\partial x}\right|_{x=x(t),y=y(t)} \cdot \frac{dx}{dt} + \left.\frac{\partial f}{\partial y}\right|_{x=x(t),y=y(t)} \cdot \frac{dy}{dt}}$$

More succinctly,

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Example

Evaluate $\frac{df}{dt}$ at $t = 0$, provided that $f(x, y) = x^2y + 2y$, $x = t^2 + 2t$, and $y = -2t + 2$.

Case 2. Suppose f remains dependent on x and y, but each in turn now depend on two quantities: t and s.

$$
\begin{tikzcd}
 & & & f \arrow[dll] \arrow[drr] & & & \\
 & x \arrow[dl] \arrow[dr] & & & & y \arrow[dl] \arrow[dr] & \\
t & & s & & t & & s
\end{tikzcd}
$$

Q: What is $\frac{\partial f}{\partial t}$? $\frac{\partial f}{\partial s}$?

To get $\frac{\partial f}{\partial t}$, we assume s is constant. Hence, it is as if x and y are dependent on t only.

So, just like in Case 1.,

$$\boxed{\frac{\partial f}{\partial t} = \left.\frac{\partial f}{\partial x}\right|_{x=x(t,s),y=y(t,s)} \cdot \frac{\partial x}{\partial t} + \left.\frac{\partial f}{\partial y}\right|_{x=x(t,s),y=y(t,s)} \cdot \frac{\partial y}{\partial t}} \text{ or}$$

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

Likewise, to get $\frac{\partial f}{\partial s}$,

$$\boxed{\frac{\partial f}{\partial s} = \left.\frac{\partial f}{\partial x}\right|_{x=x(t,s),y=y(t,s)} \cdot \frac{\partial x}{\partial s} + \left.\frac{\partial f}{\partial y}\right|_{x=x(t,s),y=y(t,s)} \cdot \frac{\partial y}{\partial s}} \text{ or}$$

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}$$

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}$$

Example

Given that $g(x, y) = x^2y + y^2x$, $x = e^{st}$ and $y = s\cos t$, find an expression for g_t and g_s.

The general case follows the same pattern.

Theorem (Multivariable Chain Rule)

Suppose f is differentiable as a function of the quantities: $x_1, x_2, \ldots, x_n$. Further suppose these quantities vary and are differentiable with respect to m variables: $t_1, t_2, \ldots, t_m$. Then f can be viewed as a function of $t_1, t_2, \ldots, t_m$, and

$$\boxed{\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}.}$$

$$\boxed{\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial t_i}.}$$

Example

Let $u = x^2 + yz$, where $x = pr\cos\theta$, $y = pr\sin\theta$, and $z = p + r$. Find u_p , u_r and u_θ when $p = 2, r = 3$ and $\theta = 0$.

Recall: given a function of two-variables $F(x, y)$, the set of solutions to $F(x, y) = C$ is the **level curve of** F at level C.

Usually, level curves are smooth curves on $\mathbb{R}^2$. Most of the time, these curves cannot be treated as the graph of a single-variable function (i.e. the vertical line test often fails).

Crucially, however, they can be treated as graphs of single-var. functions **locally**, hence there is an **implicit function** (of either x or y) that describes the curve on certain parts.

Theorem (Implicit Differentiation I)

Suppose y is a differentiable function of x satisfying the equation $F(x, y) = C$. Then

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}.$$

(Note: The constant C <u>does not affect</u> the formula.)

Example

Find $\frac{dy}{dx}$ for the implicit differentiable function y that satifies the equation $x^3 + y^3 = 1 - 6xy$. Use this to find the slope of the implicit function's graph at the point $(x, y) = (1, 0)$.

Recall: given a function of three-variables $G(x, y, z)$, the set of solutions to $G(x, y, z) = C$ is the **level surface of** G at level C.

Usually, level surfaces are smooth surfaces on $\mathbb{R}^2$. Most of the time, these curves cannot be treated as the graph of a two-variable function.

Crucially, however, they can be treated as graphs of two-var. functions **locally**, hence there is an **implicit function** (of either x & y, y & z, or x & z) that describes the surface on certain parts.

Theorem (Implicit Differentiation II)

Suppose z is a differentiable function of x & y satisfying the equation $G(x, y, z) = C$. Then

$$\boxed{\frac{\partial z}{\partial x} = -\frac{G_x}{G_z}} \text{ and } \boxed{\frac{\partial z}{\partial y} = -\frac{G_y}{G_z}}$$

Remark

1. The constant C <u>does not affect</u> the formulas.
2. If, instead, y is a differentiable function of x and z, then

$$\frac{\partial y}{\partial x} = -\frac{G_x}{G_y} \text{ and } \frac{\partial y}{\partial z} = -\frac{G_z}{G_y}.$$

Same pattern emerges when x is a function of y & z.

Example

Consider the function $H(x, y, z) = 2xyz + 2\cos(x + y)$.

1. To which level surface of H does the point $(x, y, z) = (1, -1, 3)$ belong?
2. It is known that there is a function z of x and y that locally describes the above level surface near $(1, +96 - 1, 3)$. Calculate $\frac{\partial z}{\partial x}$ at this point.
3. It is known that there is a function x of y and z that locally describes the above level surface near $(1, -1, 3)$. Calculate $\frac{\partial x}{\partial y}$ at this point.