Lec 1.6. Directional Derivatives, Gradients, and Tangent Lines/Planes

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Objectives

- To determine how to get the rate of change of a function along a particular direction
- To be familiar with the gradient and how it is used in multivariable functions equations
- To find equations for the tangent planes (resp., lines) of level surface (resp., curves)

Let us recall our problematic definition for the derivative of $f(x, y)$ at (a, b) from Lecture 1.3:

$$\lim_{(x,y)\to(a,b)} \frac{f(x, y) - f(a, b)}{\sqrt{(x-a)^2 + (y-a)^2}}$$

We remarked that this limit does not exist for most functions $f(x, y)$. Q: Why?

A: Because for a function of two variables, their graphs <u>do not have a single slope</u>; their slope changes depending on the direction along the input space.

Let $(a, b) \in \mathbb{R}^2$.

Let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector ($\sqrt{u_1^2 + u_2^2} = 1$) lying on our input space whose tail is at (a, b).

Goal: Given a function $f(x, y)$ defined near (a, b), get its slope at (a, b) along the direction of $\vec{u}$.

Ideas: Given $h > 0$,

1. $h\vec{u} = \langle hu_1, hu_2 \rangle$ is a vector of length h in the direction of $\vec{u}$
2. $(a + hu_1, b + hu_2)$ is the point that is h units away from (a, b) in the direction of $\vec{u}$.

Definition

The **directional derivative** of f along the unit vector $\vec{u}$ at (a, b), denoted by $D_{\vec{u}}(a, b)$, is defined by the limit

$$\boxed{D_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.}$$

It is the rate of change or slope of f at (a, b) along the direction of $\vec{u}$.

Example

Calculate the rate of change of the function $f(x, y) = x - y$ at the point $(2, 3)$ along the direction of the vector $\langle 2, 1 \rangle$.

Remarks

$$D_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}.$$

1. $\vec{u}$, by definition, must be a unit vector
2. If $\vec{u} = \langle 1, 0 \rangle$, then $D_{\vec{u}}(a, b) = f_x(a, b)$
3. If $\vec{u} = \langle 0, 1 \rangle$, then $D_{\vec{u}}(a, b) = f_y(a, b)$
4. Let $x(h) = a + hu_1$ and $y(h) = b + hu_2$, then

$$D_{\vec{u}}(a, b) = \frac{d}{dh} f(x(h), y(h)) \Big|_{h=0},$$

which can be further simplified.

Definition

The **gradient** of the two-variable function f, denoted by ∇f, is the function that sends a point (x, y) to the vector

$$\boxed{\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle}.$$

Similarly, one can also define the gradient of a three-variable function G:

$$\boxed{\nabla G(x, y, z) = \langle G_x(x, y, z), G_y(x, y, z), G_z(x, y, z)\rangle}$$

In either case, it is essentially just a list of the partial derivatives of the function.

As a consequence, using the dot product, the directional derivative can be rewritten as

$$D_{\vec{u}}(a,b) = \langle f_x(a,b), f_y(a,b)\rangle \cdot \langle u_1, u_2\rangle$$
$$= \boxed{\nabla f(a,b)\cdot \vec{u}}$$

Remember: the vector $\vec{u}$ has to be of **unit length**!

Example

1. Determine the directional derivative of $g(x,y) = ye^x$ at the point $(0,2)$ in the direction of $\sqrt{3}\hat{i} - \hat{j}$.
2. Find the rate of change of $h(x,y) = x^2 - 4xy + y^2$ at the point $P(0,-1)$ in the direction of $\vec{u}$, where u is the unit vector from P to $(1,2)$.

Remark

We can also talk about the directional derivative of a three-variable function G at the point (a, b, c) along the unit vector $u = \langle u_1, u_2, u_3 \rangle$:

$$D_{\vec{u}}(a, b, c) = \lim_{h \to 0} \frac{G(a + hu_1, b + hu_2, c + hu_3) - G(a, b, c)}{h}$$

It is the rate of change of G in the direction of $\vec{u}$.

Equivalently, $\boxed{D_{\vec{u}}(a, b, c) = \nabla G(a, b, c) \cdot \vec{u}}$

Example

Find the directional derivative of $F(x, y, z) = x^2y - xy^2 + z^3$ at the point $(1, 1, 2)$ in the direction of $\langle 2, 4, 4 \rangle$.

"Given a function $f(x, y)$ and a point $(a, b) \in \operatorname{dom} f$, along which direction at (a, b) is the rate of change highest (or lowest)? And by how much?"

Mathematically, we want to maximize/minimize $D_{\vec{u}}(a, b)$ over all possible unit vectors whose tails are at (a, b).

Idea: Given two/three-dimensional vectors $\vec{v}_1$ and $\vec{v}_2$, we recall that

$$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \|\vec{v}_2\| \cos\theta,$$

where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$ when their tails are joined together.

We conclude:

1. The rate of change at (a, b) is highest when $\vec{u}$ is along the direction of $\nabla f(a, b)$ with value $\|\nabla f(a, b)\|$
2. The rate of change at (a, b) is lowest when $\vec{u}$ is along the direction of $-\nabla f(a, b)$ with value $-\|\nabla f(a, b)\|$.

Example

Find the maximum and minimum directional derivatives of the following functions at the given points. Give a vector along which these values are attained.

1. $g(x, y) = ye^x$; $P(0, 2)$
2. $h(x, y) = x^2 - 4xy + y^2$; $P(0, -1)$

Given a three-var. function G and 3D unit vector $\vec{u}$, similar results are also true.

1. The rate of change at (a, b, c) is highest when $\vec{u}$ is along the direction of $\nabla G(a, b, c)$ with value $\|\nabla G(a, b, c)\|$
2. The rate of change at (a, b, c) is lowest when $\vec{u}$ is along the direction of $-\nabla G(a, b, c)$ with value $-\|\nabla G(a, b, c)\|$

Example

Find the maximum and minimum rates of change of the function $F(x, y, z) = x^2y - xy^2 + z^3$ at the point $(1, 1, 2)$ and the directions along which these values are attained.

Level surfaces and curves, revisited

Consider the level surface $G(x, y, z) = C$.

Suppose an arbitrary space curve $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$ is travelling along the surface. Then

$$G(x(t), y(t), z(t)) = C.$$

What can we conclude by differentiating both sides?

Recall

1. Given two/three-dimensional nonzero vectors $\vec{v}_1$ and $\vec{v}_2$, if $\vec{v}_1 \cdot \vec{v}_2 = 0$, then $\vec{v}_1$ and $\vec{v}_2$ are perpendicular.
2. The derivative of $\vec{R}(t)$ is always tangent to the path of the curve.

The gradient of G, when nonzero, is **normal/perpendicular** at each point on the surface $G(x, y, z) = C$.

Therefore, at any point (a, b, c) on this surface, the vector $\nabla G(a, b, c)$ is normal to the level surface at (a, b, c).

Hence, given a point (a, b, c) on the surface $G(x, y, z) = C$:

1. $\langle P, Q, R\rangle = \nabla G(a, b, c)$ is normal to the surface at (a, b, c)
2. **Equation of normal line to the surface at** (a, b, c):

$$\boxed{\vec{\ell}(t) = \langle Pt + a, Qt + b, Rt + c\rangle}$$

3. **Equation of the tangent plane to the level surface** $G(x, y, z) = C$ **at** (a, b, c)

$$\boxed{P(x - a) + Q(y - b) + R(z - c) = 0}$$

Example

Find an equation for the tangent plane and normal line to the surface defined by $zx^2 - xy^2 - yz^2 = 18$ at the point $(0, -2, 3)$.

Remark

So far, we now have two equations for tangent planes:

1. To get the tangent plane to the **graph of the function** $z = f(x, y)$ **at** $(a, b, f(a, b))$**:**

$$\boxed{z = f_x(a, b)(x - a) + f_y(a, b)(y - a) + f(a, b)}$$

2. To get the tangent plane to the **level surface** $G(x, y, z) = C$ **at** (a, b, c)**:**

$$\boxed{G_x(a, b, c)(x - a) + G_y(a, b, c)(y - b) + G_z(a, b, c)(z - c) = 0}$$

A similar argument can be made for level curves.

Given a point (a, b) on the level curve $F(x, y) = C$:

1. $\langle P, Q \rangle = \nabla F(a, b)$ is normal to the curve at (a, b).
2. **Equation of the tangent line to the level curve** $F(x, y) = C$ **at** (a, b)

$$\boxed{F_x(a, b)(x - a) + F_y(a, b)(y - b) = 0}$$

Example

Find an equation for the tangent line of the curve $x^3 + y^3 = 1 - 6xy$ at the point $(1, 0)$.

Remark

So far, we now have two equations for tangent lines:

1. To get the tangent line to the **graph of the function** $y = f(x)$ **at** $(a, f(a))$**:**

$$\boxed{y = f'(a)(x - a) + f(a)}$$

2. To get the tangent line to the **level curve** $F(x, y) = C$ **at** (a, b)**:**

$$\boxed{F_x(a, b)(x - a) + F_y(a, b, c)(y - b) = 0}$$

Extra Information

Consider the LLA of $f(x, y)$ at (a, b):

$$\begin{aligned} L_{(a,b)}(x, y) &= f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b) \\ &= \langle f_x(a, b), f_y(a, b)\rangle \cdot \langle x - a, y - b\rangle + f(a, b) \\ &= \nabla f(a, b) \cdot (\langle x, y\rangle - \langle a, b\rangle) + f(a, b) \end{aligned}$$

Compare this with the LLA of $f(x)$ at a:

$$L_a(x) = f'(a)(x - a) + f(a)$$

In this sense, the gradient is a generalization of the derivative.

Seatwork 4 (Take Home)

Consider the equation $xy^2 + yx^2 = 5 - xz^3$. It is known that there is a function $z(x, y)$ that satisfies this equation locally at the point $P(1, 2, -1)$.

1. Using implicit differentiation, evaluate $z_x(1, 2)$ and $z_y(1, 2)$.
2. Suppose x and y are parametrized via $x = e^{3uv}$ and $y = 6u \sin v + 2$. Using the chain rule and the previous item, evaluate $\frac{\partial z}{\partial v}$ at $(u, v) = (1, 0)$.
3. Give an equation for the tangent plane to the surface defined by the equation in the given at the point P.