Lec 2.1. Relative Extrema of Functions of Two Variables

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Objectives

- To know how to find the optimal values of a function given no constraints

Recall: Let $f(x)$ be defined on the interval (a, b) and let $x_0 \in (a, b)$

1. If $f(x_0) \leq f(x)$ for all x on an interval containing x_0, we say f has a **relative minimum** at x_0
2. If $f(x_0) \geq f(x)$ for all x on an interval containing x_0, we say f has a **relative maximum** at x_0

On either case, either $f'(x_0)$ does not exist or is equal to zero (i.e. x_0 is a **critical point**)

Recall: (SDT) Let $f(x)$ be twice-differentiable on the interval (a, b) and let $x_0 \in (a, b)$. Suppose $f'(x_0) = 0$.

- If $f''(x_0) > 0$, then f has a **relative minimum** at x_0.
- If $f''(x_0) < 0$, then f has a **relative maximum** at x_0.
- If $f''(x_0) = 0$, then we cannot conclude anything.

Intuitively, this works as $f(x_0) - f(x) \approx -\frac{f''(x_0)}{2}(x - x_0)^2$ when x is near x_0 (via Taylor series). I.e., f is like a parabola near x_0.

Definition

Let f be a two-variable function and let $(x_0, y_0) \in \operatorname{dom} f$. We say that, at (x_0, y_0), the function f has a

- **relative/local maximum** if there is an open disk D containing (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for any $(x, y) \in D$
- **relative/local minimum** if there is an open disk D containing (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for any $(x, y) \in D$

A **relative/local extremum** refers to either a relative maximum or a relative minimum.

Q: How do we find relative extrema?

Definition (cont'd.)

A point (x_0, y_0) contained in a disk $D \subseteq \operatorname{dom} f$ is said to be a **critical point** if one of the following is true:

- Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist, or
- $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ (i.e. $\nabla f(x_0, y_0) = \langle 0, 0 \rangle$).

Relative extrema are located at critical points, but not all critical points yield relative extrema. A critical point with no relative extremum is called a **saddle point**.

Example

Consider the function $f(x, y) = x^2 - y^2$ at $(x_0, y_0) = (0, 0)$.

Suppose $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

Goal: Check if there is a relative extremum at (x_0, y_0).

Idea: If there is a relative extremum at (x_0, y_0), all points **near** it should have function values that are larger/smaller than $f(x_0, y_0)$.

Problem: When (x, y) is near (x_0, y_0),

$$f(x, y) \approx 0(x - x_0) + 0(y - y_0) + f(0, 0) = f(0, 0)$$

which is not really useful...

Idea: If (x, y) is near (x_0, y_0), then (x_0, y_0) **is near** (x, y).

$$f(x_0, y_0) \approx f_x(x, y)(x_0 - x) + f_y(x, y)(y_0 - y) + f(x, y)$$

Likewise, since (x, y) is near (x_0, y_0):

- $f_x(x, y) \approx f_{xx}(x_0, y_0)(x - x_0) + f_{xy}(x_0, y_0)(y - y_0) + \underbrace{f_x(x_0, y_0)}_{=0}$
- $f_y(x, y) \approx f_{yx}(x_0, y_0)(x - x_0) + f_{yy}(x_0, y_0)(y - y_0) + \underbrace{f_y(x_0, y_0)}_{=0}$

For easier writing, let us set:

$$f_{xx}(x_0, y_0) = a \qquad f_{xy}(x_0, y_0) = b \qquad f_{yy}(x_0, y_0) = c$$

$$X = x - x_0 \qquad Y = y - y_0$$

We eventually get, when (x, y) is near (x_0, y_0),

$$f(x, y) - f(x_0, y_0) \approx \underbrace{aX^2 + 2bXY + cY^2}_{\textit{quadratic form}}.$$

Assuming $a \neq 0$, by completing the square on X,

$$aX^2 + 2bXY + cY^2 = a\left(X + \frac{b}{a}Y\right)^2 + \frac{(ac - b^2)}{a}Y^2$$

Set $D = ac - b^2$.

Second Derivative Test

Let f have continuous second-order partial derivatives on a disk containing (x_0, y_0). Suppose $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

Define $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.

- $D > 0$ and $f_{xx}(x_0, y_0) > 0 \Rightarrow$ relative minimum at (x_0, y_0)
- $D > 0$ and $f_{xx}(x_0, y_0) < 0 \Rightarrow$ relative maximum at (x_0, y_0)
- $D < 0 \Rightarrow$ saddle point at (x_0, y_0)
- $D = 0 \Rightarrow$ no conclusion

Examples

Determine and classify all the critical points of the following functions.

1. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$
2. $g(x, y) = x^3 + y^3 - 3x - 12y + 20$
3. $h(x, y) = x^4 - 2xy^2 + 4y^2 - 2$