Lec 3.4. Conservative Vector Fields

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Introduction

Recall: Given a vector field on $\mathbb{R}^3$:

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

- divergence of $\vec{F}$: $P_x + Q_y + R_z = \vec{\nabla} \cdot \vec{F}$,
- curl of $\vec{F}$: $\langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \vec{\nabla} \times \vec{F}$.

We have already seen the $\vec{\nabla}$ symbol in Unit 1.

Introduction

Recall: Given a real-valued function $\phi(x, y, z)$, which is a *scalar field* on $\mathbb{R}^3$, its **gradient** is given by

$$\vec{\nabla}\phi(x, y, z) = \langle \phi_x(x, y, z), \phi_y(x, y, z), \phi_z(x, y, z) \rangle,$$

which is a *vector field* on $\mathbb{R}^3$.

We can think of the gradient as the action of the del operator on a scalar field:

$$\vec{\nabla}\phi = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \phi = \langle \phi_x, \phi_y, \phi_z \rangle$$

Introduction

We have remarked before that the gradient is a generalization of the derivative. Using this, we can make a generalization for the **antiderivative**.

We will see, however, that the discussion surrounding this is not as clear as in Math 21.

Q: Given a "nice" vector field $\vec{F} = \langle P, Q, R \rangle$, is there always a scalar field ϕ such that $\vec{\nabla}\phi = \vec{F}$?

A: **NO.** Use Clairaut's theorem!

Example

Argue that $\vec{F}(x, y, z) = \langle x^2 y, yz^2, zx^2 \rangle$ cannot be the gradient of a scalar field on $\mathbb{R}^3$.

It turns out that not all vector fields (even those which look "nice") have an "antiderivative"!

Definition

A vector field $\vec{F}$ on $\mathbb{R}^n$ is said to be **conservative** if $\vec{F} = \vec{\nabla}\phi$ for some scalar field ϕ in $\mathbb{R}^n$. In this case, we call ϕ a **potential function** for $\vec{F}$.

We can extend the idea in the previous example to obtain a useful way to check if a vector field on $\mathbb{R}^3$ is conservative.

Theorem

Suppose the components of $\vec{F} = \langle P, Q, R \rangle$ have continuous first partial derivatives. If $\vec{F}$ is conservative, then $\operatorname{curl} \vec{F} = \vec{0}$.

Consequence: $\boxed{\operatorname{curl} \vec{F} \neq \vec{0} \Longrightarrow \vec{F} \text{ is not conservative!}}$

Remark

1. Our work in the previous theorem tells us that

$$\operatorname{curl}(\nabla\phi) = \vec{0},$$

where ϕ is a scalar field on $\mathbb{R}^3$.

2. We could use the ideas in the previous theorem to show that

$$\operatorname{div}(\operatorname{curl}\vec{F}) = 0.$$

Example

Explain why $\vec{F}(x, y, z) = \langle zxe^y, xze^x, xye^x \rangle$ cannot possibly be conservative.

If, however, there is reason to believe $\vec{F}$ is conservative (ex. if $\operatorname{curl} \vec{F} = 0$), then we can recover a potential function for $\vec{F}$ through a sequence of indefinite integration.

Example

Verify that $\vec{F}(x, y, z) = \langle 2xy + 1, x^2 + 2yz, y^2 - 2 \rangle$ is conservative by finding a potential function for $\vec{F}$.

FACT: If $\vec{F}$ is conservative and ϕ is a potential function for $\vec{F}$, then any other potential function for $\vec{F}$ is of the form $\phi + C$ for some constant C (similar to antiderivatives in Math 21).

Example

Show that $\vec{F}(x, y, z) = \langle 6x - 4y, z - 4x, y - 8z \rangle$ is conservative and give all its potential functions.

Example

Find all potential functions for $\vec{F}(x, y) = \langle ye^x - y, e^x - x + 5 \rangle$.

So far, we know that if $\vec{F}$ is conservative, then curl $\vec{F} = 0$.

Q: Does the converse hold?

A: The full answer is a bit complicated (will be discussed eventually). But here's a partial answer:

Fact: If the components of $\vec{F} = \langle P, Q, R \rangle$ have continuous first partial derivatives *everywhere* in $\mathbb{R}^3$, and if we also know that curl $\vec{F} = 0$, then F is conservative.

Example

Without computing potential functions, explain why $\vec{F}(x, y, z) = \langle 6x - 4y, z - 4x, y - 8z \rangle$ is conservative.

Our results also work for vector fields on $\mathbb{R}^2$. Given $\vec{F}(x, y) = \langle P, Q\rangle$, recall that curl $\vec{F} = \langle 0, 0, Q_x - P_y\rangle$:

1. If $Q_x \neq P_y$, then $\vec{F}$ is not conservative.
2. If P and Q have continuous partial derivatives and $Q_x = P_y$ *everywhere* in $\mathbb{R}^2$, then $\vec{F}$ is conservative.

Example

Without computing potential functions, explain why $\vec{F}(x, y) = \langle ye^x - y, e^x - x + 5\rangle$ is conservative.

Q: Why study conservative vector fields? Why are they called "conservative"? What do they conserve?

A: To be answered at Unit 4.

END.

Seatwork 12

Let $\vec{F}(x, y, z) = \langle 2x \cos y, e^z - x^2 \sin y, ye^z \rangle$ be a vector field defined on $\mathbb{R}^3$.

1. Without computing potential functions, **properly** explain why $\vec{F}$ is conservative.
2. Find all potential functions of $\vec{F}$.
3. Calculate $\vec{\nabla}(\vec{\nabla} \cdot \vec{F})$.

BONUS. (2 points) Let $\vec{F} = \langle P, Q, R \rangle$, where P, Q, and R have continuous second-order partial derivatives. Prove that $\text{div}(\text{curl}\, \vec{F}) = 0$.