Lec 3.6. Line Integrals of Scalar Fields

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Introduction

Let us now go back to (not triple) integrals! But now, we will be integrating over **curves**.

Motivation: Let C be a smooth curve on $\mathbb{R}^2$ and let $f(x, y)$ be a positive function defined on C. What is the area of the "curtain" made by $f(x, y)$ and C?

Idea: Divide C into small almost-straight segments $\{\Delta s_i\}_{i=1}^{n}$. Pick a point (x_i, y_i) from each Δs_i, so that the strip lying on Δs_i has area approximately $f(x_i, y_i) \cdot \Delta s_i$.

The total approximate area is $\sum_{i=1}^{n} f(x_i, y_i) \cdot \Delta s_i$. This is expected to become more accurate as $n \to \infty$.

Definition

Let f be a real-valued function on $\mathbb{R}^2$ defined on a smooth curve C. The **line integral of f along C with respect to arclength** is defined as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\cdot \Delta s_i.$$

FACT: If C can be parametrized by the vector-valued function $\vec{R}(t) = \langle x(t), y(t)\rangle$ for $t \in [a,b]$, then

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\|\vec{R}'(t)\|\,dt}.$$

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Example

Let $f(x,y) = x^2y + x$ and let C be the curve traced by $\vec{R}(t) = \langle 3\sin t, 3\cos t\rangle$, $t \in [0, \frac{\pi}{2}]$. Evaluate $\int_C f(x,y)\, ds$.

Remarks

- A curve C can be traced by many different $\vec{R}$'s. Even so, $\int_C f(x,y)\,ds$ is still the area of a **fixed** region, and hence does not change regardless of the choice of $\vec{R}$. That is, if $\vec{R}_1$ and $\vec{R}_2$ both trace C, then

$$\int_a^b f(\ldots)\|\vec{R}_1'(t)\|\,dt = \int_a^b f(\ldots)\|\vec{R}_2'(t)\|\,dt$$

- If C is a union of smooth curves $C_1, C_2, \ldots, C_n$, then

$$\boxed{\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \ldots + \int_{C_n} f(x,y)\,ds}$$

Some useful parametrizations:

- To parametrize a line segment from point P to point Q:

$$\vec{R}(t) = (1-t)P + tQ, \quad t \in [0,1]$$

- To parametrize the graph of $y = g(x)$ from $x = a$ to $x = b$:

$$\vec{R}(t) = \langle t, g(t) \rangle, \quad t \in [a,b].$$

Example

Evaluate $\displaystyle\int_C 2x\,ds$, where C consists of the portion of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segment from $(1,1)$ to $(3,2)$.

Some Applications: Mass

Suppose C models a wire on $\mathbb{R}^2$ with $\underbrace{\text{mass density}}_{mass/length}$ $\delta(x, y)$:

- mass of wire: $M = \displaystyle\int_C \delta(x, y)\, ds$
- center of mass of wire: $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{M}\int_C x \cdot \delta(x, y)\, ds, \qquad \bar{y} = \frac{1}{M}\int_C y \cdot \delta(x, y)\, ds$$

Example

Find an expression for the mass of a circular wire of radius 2 on the xy-plane with mass density $\delta(x, y) = x^2 + 3y^2$.

For reasons we will know in Unit 4, we are interested in the following integrals:

Definition

Given f defined on the smooth curve $C \subseteq \mathbb{R}^2$ parametrized by $\vec{R}(t) = \langle x(t), y(t) \rangle$ for $t \in [a, b]$, we shall define

$$\boxed{\int_C f(x,y)\, dx := \int_a^b f(x(t), y(t))\, x'(t)\, dt}$$

$$\boxed{\int_C f(x,y)\, dy := \int_a^b f(x(t), y(t))\, y'(t)\, dy}$$

We call these the **line integrals along** C **with respect to x and y**, respectively.

Remarks

Unlike line integrals with respect to arclength, these line integrals may change depending on how $\vec{R}$ traces C.

In particular, if C is a curve traced by $\vec{R}$ and $-C$ is the curve traced in the opposite manner to $\vec{R}$, then

$$\int_{-C} f(x,y)\,dx\ (or\ dy) = -\int_{C} f(x,y)\,dx\ (or\ dy).$$

Also, if C is the union of smooth curves C_1, C_2, ..., C_n,

$$\int_{C} f(x,y)\,dx = \int_{C_1} f(x,y)\,dx + \ldots + \int_{C_n} f(x,y)\,dx$$

The above formulas remain true when dx is replaced with dy.

In a future lesson, we will often combine integrals with respect to x and y, so as a shorthand notation,

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy =: \int_C P(x,y)\,dx + Q(x,y)\,dy$$

Example

Let C be the curve parametrized by $\vec{R}(t) = \langle 1 - t^2, t \rangle$ where $t \in [-1,2]$. Evaluate $\displaystyle\int_C 3y\,dx + 4xy\,dy$.

We can also extend these notions of line integrals to 3D space.

Suppose C is a smooth curve on $\mathbb{R}^3$ parametrized by $\vec{R}(t) = \langle x(t), y(t), z(t) \rangle$ for $t \in [a, b]$. Suppose $f(x, y, z)$ is defined on C. Then we have

$$\boxed{\int_C f(x, y, z)\, ds := \int_a^b f(x(t), y(t), z(t))\, \|\vec{R}'(t)\|\, dt}$$

$$\boxed{\int_C f(x,y,z)\,dx := \int_a^b f(x(t),y(t),z(t)) \cdot x'(t)\,dt}$$

$$\boxed{\int_C f(x,y,z)\,dy := \int_a^b f(x(t),y(t),z(t)) \cdot y'(t)\,dt}$$

$$\boxed{\int_C f(x,y,z)\,dz := \int_a^b f(x(t),y(t),z(t)) \cdot z'(t)\,dt}$$

At this point, it seems that we are just making these up for nothing. But we will see their applications in Unit 4.

Example

1. Evaluate $\int_C x^2 \sin z \, ds$, where C is the curve parametrized by $\vec{R}(t) = \langle \cos t, \sin t, t \rangle$, $t \in [0, \pi]$.
2. Let C be the line segment from the point $P(3, 0, -1)$ to $Q(2, -3, 1)$. Evaluate $I = \int_C (y - x) \, dx + z \, dy + (x - y) \, dz$.

Seatwork 13 (Due on July 14)

1. Evaluate $\int_C y\,dx + z\,dy - x\,dz$, where C is the line segment from $(0, 1, 2)$ to $(1, 3, 6)$ followed by the line segment from $(1, 3, 6)$ to $(1, 3, 2)$.
2. Evaluate $\int_C xy + y^2\,ds$, where C is the lower half of the circle $x^2 + y^2 = 9$ described in the counter-clockwise direction.
3. Set-up an integral with respect to t that gives the mass of a thin wire in the form of the curve $x = 2t$, $y = \ln t$, and $z = 4\sqrt{t}$ if the density function is $\delta(x, y, z) = \sqrt{x + y}$