Lec 4.4. Surface Integrals of Scalar Fields

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Introduction

So far, in double integrals, we have only been integrating over flat surfaces in the xy-plane or the uv-plane. Today, we will **integrate functions over surfaces** that are not necessarily flat.

Motivation

Let $\mathcal{S}$ be a curved lamina S, with nonuniform density. In particular, if (x, y, z) is a point in S, the mass density (mass/area) near (x, y, z) is given by $f(x, y, z)$. What is the mass of $\mathcal{S}$?.

Idea: Divide S into small pieces, say $\{\sigma_i\}_{i=1}^{n}$. Let $\Delta\sigma_i$ be the area of σ_i and (x_i, y_i, z_i) be a point in σ_i. Then the mass is approximately

$$\sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta\sigma_i$$

By letting $n \to \infty$, we expect this approximation to get closer to the actual mass of $\mathcal{S}$.

Definition

The **surface integral** of the scalar field $f(x, y, z)$ that is continuous over a smooth surface $\mathcal{S}$ is given by

$$\iint_{\mathcal{S}} f(x, y, z)\, d\sigma = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta\sigma_i,$$

provided the limit exists.

As always, we will not use surface integrals via the definition.

Let S be a 3D surface parametrized by $\vec{R}(u, v)$, $u, v \in D$. Suppose $f(x, y, z)$ is a function defined on the points of S.

Intuitively, every small rectangle in D gets sent by $\vec{R}$ to another small piece of $\mathcal{S}$ that resembles a parallelogram.

If $\Delta\sigma$ is the area of the small piece in $\mathcal{S}$, then

$$\Delta\sigma \approx \|\vec{R}_u(u, v) \times \vec{R}_v(u, v)\| \Delta u \, \Delta v.$$

Thus, the mass of this small piece of $\mathcal{S}$ is

$$\boxed{f(\vec{R}(u, v)) \cdot \|\vec{R}_u(u, v) \times \vec{R}_v(u, v)\| \Delta u \, \Delta v}$$

Theorem

If $\mathcal{S}$ is parametrized by $\vec{R}(u,v)$, then

$$\boxed{\iint_{\mathcal{S}} f(x,y,z)\,d\sigma = \iint_{D} f(\vec{R}(u,v))\,\|\vec{R}_u \times \vec{R}_v\|\,du\,dv.}$$

Remark

Quite a **lot** of quantities have to be prepared to just to evaluate this integral, so be careful in constructing the integral.

$$\boxed{\iint_{\mathcal{S}} f(x,y,z)\, d\sigma = \iint_D f(\vec{R}(u,v))\, \|\vec{R}_u \times \vec{R}_v\|\, du\, dv.}$$

Example

1. Evaluate the surface integral $\iint_S x + z\, d\sigma$, where S is the surface parametrized by

$$\vec{R}(u,v) = \langle u, 3\cos v, 3\sin v\rangle,\ 0 \le u \le 4,\ 0 \le v \le \frac{\pi}{2}.$$

2. Evaluate $\iint_{\mathcal{S}} 2x - 3y + z\, d\sigma$, where $\mathcal{S}$ is parametrized by

$$\vec{R}(u,v) = \langle 2v, v-u, u\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2-2u.$$

Recall: A function $z = g(x, y)$ defined on a region R can be parametrized as

$$\vec{R}(x, y) = \langle x, y, g(x, y)\rangle, (x, y) \in R.$$

Using this, if $\mathcal{S}$ is the surface determined by the graph of $z = g(x, y)$ on R, then

$$\boxed{\iint_{\mathcal{S}} f\, d\sigma = \iint_R f(x, y, g(x, y))\sqrt{1 + g_x(x, y)^2 + g_y(x, y)^2}\, dA}$$

Remark

Note that g is associated with the surface, while f is associated with the integrand. Also, the double integral on the right must be **in terms of x and y only! No z!**

$$\boxed{\iint_{\mathcal{S}} f\,d\sigma = \iint_R f(x,y,g(x,y))\sqrt{1+g_x(x,y)^2+g_y(x,y)^2}\,dA}$$

Example

1. Let $\mathcal{S}$ be the portion of the plane $2x+2y+z=4$ in the first octant. Find the mass of $\mathcal{S}$ provided it has mass density given by $f(x,y,z)=x+z$.
2. Evaluate $\displaystyle\iint_S z^2\,d\sigma$, where S is the portion of the sphere $x^2+y^2+z^2=4$ above $z=1$.

Remark

We have **two** formulas for the surface integrals of scalar fields. They have their own use cases.

1. When $\mathcal{S}$ is a surface parametrized by $\vec{R}(u,v)$, $(u,v) \in D$,

$$\boxed{\iint_{\mathcal{S}} f\, d\sigma = \iint_{D} f(\vec{R}(u,v))\, \|\vec{R}_u \times \vec{R}_v\|\, du\, dv.}$$

2. When $\mathcal{S}$ is the graph of $z = g(x,y)$, $(x,y) \in D$,

$$\boxed{\iint_{\mathcal{S}} f\, d\sigma = \iint_{R} f(x,y,g(x,y)) \sqrt{1 + g_x(x,y)^2 + g_y(x,y)^2}\, dA}.$$

   Remember that the integral on the right side should not have any z!

For next lesson...

Today we have discussed how to integrate real-valued functions on surfaces. Next time, we will be integrating vector fields over surfaces, using the concept of **flux**.

Recall: Given a flat surface S with area A and unit normal vector $\hat{n}$ and a constant vector field $\vec{F}$ acting on S, the **flux** of $\vec{F}$ across S is defined as

$$(\vec{F} \cdot \hat{n})A$$

VERY roughly speaking, if $\vec{F}$ measures the strength and direction of movement of a fluid, the flux $(\vec{F} \cdot \hat{n})A$ measures how much of that fluid is going through the surface S along the direction of $\hat{n}$.

Example

A constant vector field $\vec{F} = 8\hat{i} - 4\hat{j} + 2\hat{k}$ is acting on a flat surface S with an area of 10 $units^2$ and normal vector $\vec{n} = \langle 1, 2, 1 \rangle$. What is the flux of $\vec{F}$ across S in the direction of $\vec{n}$?

Remarks

When $\vec{F}$ is constant and the surface is flat,

1. Positive flux: $\vec{F}$ has a component that flows along $\vec{n}$.
2. Negative flux: $\vec{F}$ has a component that flows opposite of $\vec{n}$.
3. Zero flux: $\vec{F}$ moves tangent to the surface.

Next time, we will consider the general problem of finding the flux of a <u>varying</u> vector field across a <u>curved</u> surface.